Stable great ratios and stationary dispersion of profit rates in unbalanced growth*

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abstract
Trajectories of ongoing price change, unbalanced growth, and nonneutral technical progress directed by the profit-driven production and innovation decisions of capitalists display two outstanding macroscopic features of late-capitalist development—trendless ratios of the value of output to the value of capital, and trendless shares of wages and profits in social income—while supporting, as a long-run microscopic outcome of competition alternative to profit-rate equalization, stationary nondegenerate distributions of capital over an interval of profit rates.

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1 Long-run price theory and ongoing technical change

To make room for continuing technical change in the theory of relative prices and multisector growth is not easy. The most familiar and least satisfying proposal assigns a common rate of exogenously Harrod-neutral technical change to every sector, so that an equally rapid growth in the real wage leaves an economy's equilibrium prices and activity proportions unchanged (Schefold [1976]). As a basis for studying actual economies this draws the objections that rates of labor productivity growth show a great deal of variation across sectors and that innovation is far from Harrod neutral at the level of individual sectors.

Another strategy looks for laws of motion for prices and activities for which a unique price vector and balanced growth ray constitute an asymptotically stable point of rest so that prices and quantities evolve to stay within a moving neighborhood of the transient "long-run" equilibria swept out by an ongoing perturbation of the technology. Duménil and Lévy [1995b] have applied this strategy to classical "cross-dual" dynamics in two-good economies with some success. And Krause and Fujimoto [1986] have shown that a process in which output prices are marked up over input costs converges to prices of production for an arbitrary path of a nonstationary technology. But progress along these lines is limited by the dearth of economically significant stability conditions for higher-dimensional price and quantity motions that allow for realistic interactions between the prices and the quantities (Nikaido [1985]).

The alternative that I'll try out in this paper is to revise the conception of long-run equilibrium so as to *build in* endogenous technical change while permitting its profile to vary from sector to sector.

I'll use a continuous-time model of the production of n goods by means of those goods and homogeneous labor. As in Salvadori [1998] capitalists operate production activities that map vectors of produced input stocks and nonproduced labor flows into single-good production flows. Innovation takes place on and continually displaces a *frontier* containing those activities that are the most profitable in current prices. Capitalists on this frontier choose proportional rates of change in their activities' input-output coefficients from a convex innovation set so as to maximize their profit rates' instantaneous rates of change in current prices, and each capitalist's constrained-best direction of innovation is given by the vector of the different inputs' current shares in her total costs.

I discuss a class of trajectories for prices and the social technology that were first explored by Kennedy [1972] (and see also Craven [1973] and Orosel [1977]) and that I'll call *quasi-neutral.* On these paths prices and frontier input-output coefficients change at constant but generally unequal proportional rates. Labor productivity is growing in each sector but at rates that differ across sectors, and nonlabor input requirements are variously increasing or decreasing. Though they violate the stationarity that is built into better-known conceptions of long-run equilibrium, these price motions and innovation directions maintain a constant uniform profit rate on the frontier, and they stabilize inputs' shares in the frontier capitalists' costs at the unique configuration that directs the capitalists to opt for quasi-neutrality.

These paths agree with two striking patterns of actual structural change: the divergence of labor productivities across sectors that I mentioned before, and the Salter [1960] pattern of relative prices that move in a direction opposite to these movements of relative labor productivities.

The corresponding market-clearing trajectories of production satisfy a condition that I'll call *value-balanced growth*. Though physical input stocks and output flows diverge, their values in current prices follow a common exponential path. Though I don't claim that actual growth is value-balanced, this seems in any case a better approximation than balanced growth. Balanced growth is in fact decisively rejected, for example in Whelan's study [2004] of US NIPA data; Whelan finds that while investment and consumption goods account for stationary shares of *nominal* output, a divergence in their physical quantities has been offset by a long-term decline in the relative price of capital goods. Value-balanced growth under quasi-neutral innovation also implies that rates of sectoral output growth are increasing in sectors' rates of labor productivity growth, a pattern noticed in the data by Fabricant [1940] and many later researchers (Metcalfe, Foster, and Ramlogan [2005]).

To a growth theorist's eye these value-balanced paths stand out for their aggregate profile, which is pure Kaldor: constant value-of-output/value-of-capital ratios, constant class income shares, and constant growth rates for these value aggregates and for employment-weighted aggregate labor productivity.[1]

This idealization repays attention for a second reason. By following a quasi-neutral innovation path over time the economy of this paper uncovers a continuum of activities for producing each of the goods. And the profitability of operating an activity from one of these technological *lineages* is described by a time-invariant decreasing function of the age of the activity.

Say that a *capital distribution* assigns to each interval of profit rates that proportion of the mass of value of capital that's tied up in activities whose profitabilities lie in that interval. I will show that for any small-enough growth rate of the social mass of value of capital there exists a capital distribution that reproduces itself under a dynamics of slow capital reallocation regulated by relative profitability, with the mass so distributed expanding at the given rate. So the model's aggregate value-balanced growth paths can be instantiated in the small by a stationary capital distribution.

In effect this argument takes the advice of Farjoun and Machover [1983], reopening the question of an appropriate competitive boundary condition for long-run prices. In actual capitalist economies the lion's share of productive capacity is given over to activities that earn far less than the maximum observed profit rate. Rather than impose equal profitability on all the operated activities as in the classical price theory surveyed by Kurz and Salvadori [1995], the paper shows that deterministic profit rate *distributions* are supported by a gradual adjustment of capital toward higher returns. Profit rate equalization reappears here in the less restrictive form of uniform frontier profitability in the different sectors.

These stationary distributions are also a multisector, continuous-innovation cousin of the standing-wave solutions of one-sector, discrete-innovation models studied in Iwai [2000], Henkin and Polterovich [1991], and Franke [2000]. They depend for their existence on the unchanging profile of profitability's decay along the lineage of discovered activities, an invariance made possible by the quasi-neutrality of innovation and price changes to which I now turn.

2 Quasi-neutral innovation

Suppose to begin that all production is on the frontier. A production activity is described by a vector $a_j(\tau) = (a_{0j}(\tau), a_{1j}(\tau), a_{2j}(\tau), ..., a_{nj}(\tau))$ assigned to the date of its introduction τ and defined so that a

[1] Growing economies that have stationary aggregate ratios despite ongoing change in the composition of output arising variously from nonstationary consumption demand and nonstationary technology are the subject of several recent papers, including Acemoglu and Guerrieri [2005]; Ngai and Pissarides [2004]; Föllmi and Zweimüller [2002]; Kongsamut, Rebelo, and Xie [2001]. For a far more ambitious picture that takes in both forms of structural change see Pasinetti [1981].

capitalist who runs the activity while employing a flow l of labor and while holding a vector k of the n goods each yielding a flow of productive services in unit proportion to its nondepreciating stock produces a flow

$$\min\left\{\frac{l}{a_{0j}(\tau)}, \frac{k_1}{a_{1j}(\tau)}, \frac{k_2}{a_{2j}(\tau)}, ..., \frac{k_n}{a_{nj}(\tau)}\right\} \tag{1}$$

of the good j.

Capitalists continuously sell their momentary outputs, build up their input stocks with continuous purchases of produced goods, and buy labor from workers who are continuously purchasing goods for consumption in markets where a single price vector $p(t)$ rules at t; this includes the wage, $p_0(t)$, and $p_j(t)$ as the price of the jth produced good. In this section I abstract from the state of supply and demand on these markets to study certain time paths for prices and the distribution of production capacity over an evolving technology. But in section 18 I'll come back to confirm that on the corresponding trajectories of production and spending markets for produced goods are continuously cleared and a constant fraction of an exponentially expanding workforce is employed.

The ratio of the flow of profit from an activity $a_j(\tau)$ to the value of stocks tied up in it is the *profit rate*

$$r_j(\tau,t) = \frac{\max\left[p_j(t) - p_0(t)\,a_{0j}(\tau), 0\right] + \sum_{i=1} p_i(t)\,a_{ij}(\tau)\,\pi_i}{\bar{p}(t)\cdot\bar{a}_j(\tau)}, \tag{2}$$

where $\bar{p}$ and $\bar{a}_j$ are price and activity vectors with their zero*th*, labor coordinates deleted, where $\pi_i \equiv \dot{p}_i/p_i$ so that the second term in the numerator carries capital gains or losses on the holdings of input stocks, and where the first term allows capitalists to refrain from absolutely unprofitable production. As I'll explain in section 19 this profit rate controls the evolution of productive capacity since I assume that input purchases for old production activities are financed out of those activities' retained current profits.

Innovation is local in the sense that the evolution of activities obeys

$$\frac{da_{ij}(\tau)}{d\tau} = \hat{a}_{ij}(\tau)\,a_{ij}(\tau), \tag{3}$$

with $\hat{a}_j(\tau)$ a vector of proportional rates of change in the frontier production coefficients chosen at τ by the innovating capitalists from a time-invariant set of possible profiles of innovation. In particular for each j let some continuous, differentiable, strictly increasing, strictly convex function, g_j, define the set of possible innovations in the production of the jth good as those points $\hat{a}$ in $\Re^{n+1}$ for which $g_j(\hat{a}) \geq 0$. I assume that innovation possibilities are *fertile* in the sense that $g_j(0) > 0$ but *limited* in the sense that there's a $\gamma_j > 0$ such that

$$g(\hat{a}) \geq 0, \hat{a}_{jj} = 0 \Rightarrow \hat{a}_{0j} \geq -\gamma_j. \tag{4}$$

An innovating capitalist chooses $\hat{a}_j(\tau)$ in the innovation set to maximize the partial of $r(t,\tau)$ with respect to τ, which after some manipulation comes out as

$$\frac{\partial r_j(\tau,t)}{\partial\tau} = -\eta_j\cdot\hat{a}_j \tag{5}$$

where η_j is the $n+1$-vector with coordinates

$$\eta_{oj} \equiv \frac{p_0(t)\,a_{0j}}{\bar{p}(t)\cdot\bar{a}_j(\tau)} \tag{6}$$

and

$$\eta_{ij} \equiv \frac{p_i(t)\, a_{ij}}{\bar{p}(t) \cdot \bar{a}_j(\tau)} \left[r_j(t,\tau) - \pi_i \right]. \tag{7}$$

This problem's solutions are homogenous-of-degree-zero in the numbers η_j,and it will be helpful to work with the scalar multiple of them

$$\mu_j \equiv \frac{\bar{p}(t) \cdot \bar{a}_j(\tau)}{p_j(t)} \eta_j; \tag{8}$$

I'll call these last numbers *innovation weights* since $\sum_{i=0} \mu_{ij} = 1$. The innovating capitalist's problem is then

$$\min_{\hat{a}_j} \mu_j \cdot \hat{a}_j \text{ s.t. } g(\hat{a}_j) \geq 0; \tag{9}$$

let $\hat{a}_j(\mu_j)$ be the unique point in $\Re^{n+1}$ that solves it.

Now I describe a path for prices and the frontier technology on which innovating capitalists pursue an unchanging profile of innovation while earning a constant rate of profit. A stationary direction of technical change and a constant frontier profit rate are ensured if prices evolve at constant proportional rates and if for every i and j

$$\pi_j = \frac{d}{dt} \ln pa^j = \pi_i + \hat{a}_{ij}(\mu_j) \tag{10}$$

or

$$\pi_j - \pi_i - \hat{a}_{ij}(\mu_j) = 0, \quad i = 0, 1, ..., n; j = 1, 2, ..., n. \tag{11}$$

Appendix A shows that (11) has solutions. I will say that a solution $\{\pi^*, \mu^*\}$ defines a *quasi-neutral* direction of price change and technical progress.

On a quasi-neutral path

$$\pi_j^* = \mu_j^* \cdot \left(\pi^* + \hat{a}_j(\mu_j^*)\right) \tag{12}$$

while the fertility of innovation implies that for every μ_j

$$\mu_j \cdot \hat{a}_j(\mu_j) < 0, \tag{13}$$

so it must be that

$$\pi_j - \mu_j^* \cdot \pi^* < 0, \quad j = 1, 2, ..., n. \tag{14}$$

But since $\mu_j^* \cdot \pi^*$ is just a weighted average of the rates of price change, the latter inequalities require that in fact

$$\pi_0 > \pi_j, \quad j = 1, 2, ..., n; \tag{15}$$

if the weighted averages $\mu_j^* \cdot \pi^*$ are to exceed π_j for each produced good j, labor as the only *non*produced input must have a price growing faster than the rest. It follows that the real wage measured in terms of an arbitrary commodity is increasing on any quasi-neutral path. By (11), so must be each sector's labor productivity.

As Kennedy [1972] and Craven [1983] pointed out, quasi-neutrality is a kind of generalized Harrod neutrality. It permits the negative or positive augmentation of individual produced inputs. But it requires the positive augmentation of labor in every line of production and—from (11)—zero augmentation of each good in the production of its own kind.

It's especially striking that the necessity of economywide labor productivity growth—the signature of capitalist development—is an implication of stationarity in the direction of technical change alone. I mean that in deriving it I haven't made any assumptions about the shape of the innovation set, apart from the *possibility* of positive labor productivity growth. Nor have I coupled the innovation process to any specific model of income distribution, labor supply, or capital accumulation. The necessity of labor augmentation is a pure consequence of the fact that labor is the only commodity that's not produced for profit. This conclusion is a sort of technological-evolutionary correlate of the classical insight—expounded for example by Burgstaller [1994] and Foley [2003b]—that the qualitative behavior of capitalist production is shaped by the differential conditions of reproduction of its various inputs.

Notice that if some $\{\pi^*, \mu^*\}$ is a solution, so is a $\{\pi^* + a, \mu^*\}$ with a an arbitrary $n+1$-vector of identical components. Since these solutions have the same technical change profile, the same time paths for relative prices, and profit rates for all activities that differ by the common term a, I'll normalize them by pegging the rate of change of one of the prices to a given real number. In particular let $\pi_0^* = 0$ so that if the wage rate in the initial condition of a quasi-neutral path is 1, all subsequent prices are in terms of labor.

3 Value-balanced growth

If production of the jth good is increasing exponentially at g_j, purchases by the jth producers of the ith good have the growth rate

$$g_{ij} = g_j + \hat{a}_{ij} \tag{16}$$

and so on the quasi-neutral path

$$g_{ij} = g_j + \pi_j^* - \pi_i^*. \tag{17}$$

Given that price and technical changes are quasi-neutral, then, the values of the different sectors' production and input flows and input stocks in current prices are in constant proportion to one another,

$$g_j + \pi_j^* - g_i - \pi_i^* = 0, \tag{18}$$

if and only if

$$g_{ij} = g_i, \tag{19}$$

for all i and j so that the output of each good is increasing at the rate needed to supply the input requirements of all the other activities. Where innovation is quasi-neutral, market clearing implies and is implied by value-balanced growth.

Let g be the common growth rate of the value-of-output and value-of-capital masses on such a trajectory. Since $\hat{a}_{oj} = \pi_j^*$ the growth rate of production employment in each sector is also $g = g_j + \pi_j^*$. If the potential workforce grows like e^{nt}, a growth path with $g = n$ maintains a constant ratio of employed to available workers. Growth rates for the production of the produced goods that stabilize the employment ratio fall out from

$$g_j^* = n - \pi_j^*; \quad j = 1, 2, ..., n \tag{20}$$

As the money wage equals 1 forever, the economy's wage bill is also increasing at n on such a path. So if workers, the only consumers, consume their wages entirely while devoting a constant fraction of spending to each of the goods they consume, consumption purchases of the jth good also have a growth rate equal to the righthand side of (20). As befits the economy described by Fabricant, the growth rates of production in (20) are increasing in the sector-specific rates of labor-productivity growth.

I've now shown that if markets for goods and labor clear in the initial condition, they remain clear on a growth path that satisfies (20) with quasi-neutral technical and price changes and a proportional expansion of the values of input stocks and output flows. To secure that value-balanced expansion in turn, it would be enough to equalize profit rates at the frontier while assuming that investment is proportional to profitability and that capacity is costlessly adjusted to remain at the frontier. But I think it will be more interesting to allow for the fact that most production is subprofitable.

4 Slow capacity adjustment

I'll now suppose that productive capacity, rather than being piled up at the frontier, is distributed over the continuum of activities $a_j(\tau)$ that's been traced out by the innovation process. The stocks of the n produced goods are shared out between sectors and smeared over the activities within the sectors according to continuous densities $m_j(\tau)$, and the produced flow of good j is represented by integrating (1) over that continuum weighted by the $m_j(\tau)$s, a well-defined operation given the continuity of the $a_j(\tau)$s and the $m_j(\tau)$s.

I assume that profits from old activities are either plowed back to finance investment in those activities or sent to the frontier to establish capacity in the newest activities. When frontier production earns a profit rate R, spending on those activities is had by summing over j the integral over τ of

$$\xi\left(r_j(\tau,t),R\right)c_j(\tau,t)$$

where the intensity of diversion to the frontier is some function $\xi(r,R)$ that satisfies

$$0<\xi(r,R)\le r \text{ and } \frac{\partial\xi(r,R)}{\partial r}<0<\frac{\partial\xi(r,R)}{\partial R} \tag{21}$$

for $R>r>0$ and with $\xi(r,R)=0$ for $r\le 0$ and where

$$c_j(\tau,t)\equiv m_j(\tau)\,\bar{p}(t)\cdot k(t). \tag{22}$$

Let

$$C_j(\tau,t)\equiv\int_{-\infty}^{\tau}m_j(\sigma)\,\bar{p}(t)\cdot k(t)\,d\sigma,$$

report the value of the capital mass committed to j-producing activities introduced as of τ. Then for $t>\tau$ this expands according to

$$\frac{\partial C_j(\tau,t)}{\partial t}=\int_{-\infty}^{\tau}\left[r_j(\sigma,t)-\xi\left(r_j(\sigma,t),R\right)\right]c_j(\sigma,t)\,d\sigma. \tag{23}$$

This description of investment is familiar from modern-classical modelling of price and investment dynamics (Nikaido [1985], Duménil and Lévy [1993]), where it's commonly supposed that the relative growth rates of industries' capital stocks are increasing in their relative profitabilities. It's also a multisector counterpart to the single-good process of innovation diffusion via differential growth studied by Franke [2000] and Iwai [2000]. Duménil and Lévy [1998] have shown that it can be rationalized by the costs of capital adjustment that confront financially-constrained profit-maximizing investors.

Notice as well that by substituting for ξ the functions

$$\xi_j(r,R)=r-\frac{\sum_{i=1}p_i(t)\,a_{ij}(\tau)\,\pi_i}{\bar{p}(t)\cdot\bar{a}_j(\tau)} \tag{24}$$

you'd have that the productive capacity of each activity is fixed once and for all by the initial investment in it. The resulting capital distribution would resemble the outcomes of a vintage-capital model like that of Solow, Tobin, von Weizsäcker, and Yaari [1966], with capital as a function of the age of activities decaying exponentially from the newest activity. I don't make much of this, though, because it seems that actual capital distributions (for example the ones depicted in Wells [2001]) are more often hump-shaped than monotonic.

5 Stationary capital distributions

Consider an economy that's following a quasi-neutral path of price and technical change $\pi^*, \hat{a}^*$ so as to maintain a constant profit rate R on activities at the production frontier. In the terms of Anwar Shaikh's development of Marx's discussion of competition in volume 3 of *Capital*, the occupants of this frontier are "regulating capitals" whose changing technology dictates the path of prices faced by the less efficient capitals (Botwinick [1993]).

The profitability of running an activity in the jth lineage is a time-invariant function of the time $s \equiv t - \tau$ elapsed since its discovery,

$$r_j(s; R) \equiv \frac{\max[e^{\pi_j s} - \mu^*_{oj}, 0] + \sum_{i=0} e^{\pi_i s} \eta^*_{ij} \pi^*_i / (R - \pi^*_i)}{\sum_{i=0} e^{\pi_i s} \eta^*_{ij} / (R - \pi^*_i)} \tag{25}$$

which is decreasing in s and which approaches a negative limit as s approaches ∞.

Let a function

$$\Psi_j(\tau, t) \equiv C_j(\tau, t)\, e^{-gt}$$

give the proportion of total value of capital at t, e^{gt}, that's dedicated to j-producing activities introduced as of τ. I'm looking for $F_j(s)$ and θ_j that satisfy for all t

$$\Psi_j(\tau, t) = \Psi_j(t, t) - F_j(t - \tau) = \theta_j - F_j(s) \tag{26}$$

so that $F_j(s)$ is the time-invariant proportion of capital tied up in j-producing activities no older than s and θ_j is the time-invariant capital share of the jth sector. Where $F_j(s)$ exists it follows that

$$c_j(\tau, t)\, e^{-gt} = \frac{\partial \Psi_j(\tau, t)}{\partial \tau} = -\frac{\partial \Psi_j(\tau, t)}{\partial t} = \frac{dF_j}{ds} \equiv f_j(s). \tag{27}$$

With the time evolution of $C_j(t, \tau)$ given by (23) you have

$$\frac{\partial \Psi_j(\tau, t)}{\partial t} = \int_{-\infty}^{\tau} [r_j(\upsilon, t) - \xi(r_j(\upsilon, t))]\, c_j(\upsilon, t)\, e^{-gt} d\upsilon - g\Psi_j(\tau, t), \tag{28}$$

and then by (26)

$$f_j(s) = -\int_s^{\infty} [r_j(\sigma) - \xi(r_j(\sigma))]\, f_j(\sigma)\, d\sigma + g(\theta_j - F_j(s)). \tag{29}$$

Differentiating through (29) gives

$$\frac{df_j(s)}{ds} = [r_j(s) - \xi(r_j(s)) - g]\, f_j(s). \tag{30}$$

Evaluating both sides of (29) at $s = 0$ and summing over j implies that

$$\sum_j f_j(0) = \sum_j \int_0^{\infty} \xi(r_j(s))\, f_j(s)\, ds \tag{31}$$

given that

$$g - \sum_j \int_0^{\infty} r_j(s)\, f_j(s)\, ds = 0 \tag{32}$$

as all profits are reinvested and the value mass has g as its growth rate.

Functions $f_j(s)$ that solve (30) with initial conditions that satisfy (31) and normalized so as to integrate to 1 thus describe a stationary capital distribution over the age continuum of activities. And where $s_j(r)$ assigns to r the s that solves $r_j(s; R) = r$, the density

$$\phi(r) \equiv \sum_j f_j(s_j(r))$$

describes a distribution of capital by profitability that reproduces itself under the investment dynamics of the last section.

In Appendix B I show that distributions like this exist for each quasi-neutral price and technology path and for a small-enough growth rate g. The next step is to compute explicit distributions for particular numerical innovation paths and to compare these with actual profit-rate distributions like the ones in Wells [2001]. This paper has taken a first step by spelling out a many-commodity model in which reinvestment regulated by differential profitability generates self-reproducing distributions.

Appendix A. Existence of quasi-neutral price and technical evolution

Define the family of sets

$$Z_\varepsilon \equiv \left\{ \pi \in \Re^{n+1} \left| \pi_0 = 0 \text{ and } \forall j \neq 0, -\varepsilon \geq \pi_j \geq -\lambda_j \right. \right\}, \tag{33}$$

parametrized by small positive numbers ε, and the functions

$$\alpha_j(\pi) = \sup \left\{ \alpha \left| g\left(\alpha \pi^j - \alpha \pi\right) = 0 \right. \right\}, \tag{34}$$

where π^j is the $n+1$-vector with components equal to π_j. Let π_j^t obey

$$\pi_j^{t+1} = \min\left[-\varepsilon, \alpha_j\left(\pi^t\right) \pi_j^t\right], \quad j = 1, 2, ..., n \tag{35}$$

with π_0^t equal to zero for all t. By construction g is zero at the point $\alpha_j(\pi^t)\pi^{j,t} - \alpha_j(\pi^t)\pi^t$, whose jth coordinate is zero, so its 0th coordinate $\alpha_j(\pi^t)\pi_j^t$ is greater than or equal to $-\lambda_j$. So this recursion sends points of Z to points of Z. Also it's continuous as the functions $\alpha_j(\pi^t)$ are continuous. So it has a fixed point. Now suppose that for some j

$$\lim_{\varepsilon \to 0} \sup_{\pi \in Z_\varepsilon} \alpha_j(\pi)\pi_j = 0. \tag{36}$$

Then writing the π in Z_ε that gives the greatest $\alpha_j(\pi)\pi_j$ as π_ε you'd have

$$\lim_{\varepsilon \to 0} g\left(\alpha_j(\pi_\varepsilon)\pi_\varepsilon^j - \alpha_j(\pi_\varepsilon)\pi_\varepsilon\right) = g\left(-\lim_{\varepsilon \to 0} \alpha_j(\pi_\varepsilon)\pi_\varepsilon\right) \geq g(0) > 0 \tag{37}$$

violating the definition of $\alpha_j(\pi)$. So in fact $\alpha_j(\pi_\varepsilon)\pi_\varepsilon^j$ must have a negative upper bound, say $-\zeta$. Therefore by choosing $\varepsilon < \zeta$ you can ensure that a fixed point of the corresponding mapping has

$$\alpha_j(\pi^*) = 1 \tag{38}$$

for every j, from which it follows $\pi^{j*} - \pi^*$ lives on the innovation frontier. If you then put

$$\mu_j^* = \left[\sum_i \frac{\partial g_j\left(\pi^{j*} - \pi^*\right)}{\partial \hat{a}_{ij}}\right]^{-1} \frac{\partial g_j\left(\pi^{j*} - \pi^*\right)}{\partial \hat{a}_{ij}}, \tag{39}$$

so that μ_j^* is the vector of innovation weights that rationalizes innnovation in the profile $\pi^{j*} - \pi^*$, you know that there's a $\{\pi^*, \mu^*\}$ satisfying (11) with $\pi^* \leq 0$ and $\pi_0^* = 0$.

Appendix B. Existence of stationary capital distributions

Let

$$\beta_j(R) \equiv \int_0^\infty \xi(r_j(s;R), R)\, e^{-gs+\int_0^s [r_j(\sigma;R)-\xi(r_j(\sigma;R),R)]d\sigma} ds. \tag{40}$$

This is continuous in R. Since $R \geq \xi(r_j(s;R), R) \geq 0$ it has $\beta_j(0) = 0$. Since $\xi(r_j(s;R), R) > 0$ for $R > r > 0$ it has $\beta_j(R) \to \infty$ as $R \to \infty$.

Since a solution of (30) satisfies

$$f_j(s) = f_j(0)\, e^{-gs+\int_0^s [r_j(\sigma;R)-\xi(r_j(\sigma;R),R)]d\sigma} ds, \tag{41}$$

(31) is equivalent to

$$\sum_j f_j(0) = \sum_j \beta_j(R)\, f_j(0). \tag{42}$$

As R goes from 0 toward ∞, so does the righthand side of (42). So for any $f_j(0)$s there exists an R^* such that (42) is satisfied. If g is not too big, $R^* > g$. So an arbitrary vector of capital shares

$$\theta_j = f_j(0) \int_0^\infty e^{-gs+\int_0^s [r_j(\sigma;R)-\xi(r_j(\sigma;R),R)]d\sigma} ds \tag{43}$$

can be supported by choosing suitable $f_j(0)$s. So you can in particular choose them to match the capital shares required by a market-clearing initial condition of the quasi-neutral, value-balanced growth path.

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