Production decentralized by decentralized exchange

June 20, 2011

A. J. Julius
Department of Philosophy, UCLA

Abstract. Inputs to production change hands at dispersed prices in processes of decentralized reallocation. These processes converge to equilibria with uniform prices marketwide. Final patterns of production and distribution can depend on institutionally specific trade and bargaining practices. Pure exchange among firms fails to deliver efficient production or to equalize firms' returns. The efficient and return-equalizing equilibria that attract trajectories of decentralized arbitrage maximize the value of social output, whose gradient with respect to aggregate stocks coincides with equilibrium prices. When production by growth-oriented firms alternates with rounds of negotation over long-term supply contracts, the economy approaches a path of balanced expansion, and prices emerge to equalize profitability at the attracting rate of growth.

1 Getting serious

Price theory so far is a theory of equilibria. From Duncan Foley I learned to take these equilibria seriously only where there's reason to believe that they're the stable rest points of dynamical systems representing the decentralized motions of goods across persons characteristic of some scheme of persons' property in goods.

Economists have turned up one fairly general argument to that effect. Consider a process of voluntary exchange among selfish owners of resources that conserves their aggregate holdings while uncovering in time (perhaps a very long time) any unexploited gains to trade. It's shown in [1, 5, 8, 11] that every path of such a system converges to an exchange equilibrium—to some point in the continuum of Pareto-efficient allocations that Pareto-dominate the original position.

At least one of these exchange equilibria can be supported, as a "Walrasian" balance of initial-wealth-constrained supplies and demands, by budget planes that pass through the traders' initial endowments. But in almost all cases at most a finite number enjoy such support [2], and an arbitrary trajectory probably comes to rest somewhere outside this measure-zero subset of the equilibrium set [4, 6], having first redistributed wealth across traders in ways that reflect the particular institutional configuration of the exchange process. All this I learned from Foley, too [5].

That goods are conserved in these models puts production out of the question. And the traders who populate them are simply people who like goods. Unlike capitalist firms or wealthholders, they're not concerned with the profits they can take by making and selling goods or financing their manufacture. In this paper I discuss some possibilities for and some difficulties encountered in extending this argument to models of production.

2 Production for exchange

Let the $n \times m_j$ input and output matrices $\mathbf{A}_j$ and $\mathbf{B}_j$ describe m_j production and storage activities available to the jth of z firms. Each firm holds stocks of the various goods to be used as inputs and expects to sell its eventual outputs at prices $\mathbf{p}$. Given an n-vector of stocks $\mathbf{k}_j$ and an n-vector of price guesses $\mathbf{p}$ let $\mathbf{x}_j^{\mathbf{p}}(\mathbf{k}_j)$ solve

$$\max_{\mathbf{x}\in\Re_+^{m(j)}} \mathbf{p}\mathbf{B}_j\mathbf{x} \text{ s.t. } \mathbf{A}_j\mathbf{x} \leqq \mathbf{k}_j \tag{1}$$

and let $\lambda^{\mathbf{p}}{}_j(\mathbf{k}_j)$ solve

$$\min_{\lambda\in\Re_+^n} \lambda\mathbf{k}_j \text{ s.t. } \lambda\mathbf{A}_j \geqq \mathbf{p}\mathbf{B}_j. \tag{2}$$

Write the common value of these dual programs as $v_j^{\mathbf{p}}(\mathbf{k}_j)$ and, collecting the z firms' stocks in $\mathbf{k}$, collect the z value functions in $\mathbf{v}^{\mathbf{p}}(\mathbf{k})$.

3 Exchange for production

Let's talk about a trading process, represented by a family of operators $T_{\mathbf{p}}$ with parameter $\mathbf{p}$, that for given price expectations $\mathbf{p}$ sends holdings $\mathbf{k}$ to holdings $T_{\mathbf{p}}\mathbf{k}$. (For legibility's sake I'll usually drop the $\mathbf{p}$ subscript from T and the price superscripts from the symbols representing firms' production programs.) I'll accept any continuous T such that

$$T\mathbf{k} \in Z(\mathbf{k}) \equiv \left\{\mathbf{k}' \middle| \sum_j \mathbf{k}'_j = \sum_j \mathbf{k}_j\right\}: \tag{3}$$

goods are neither created nor destroyed in trade;

$$\mathbf{v}(T\mathbf{k}) \geqq \mathbf{v}(\mathbf{k}): \tag{4}$$

firms refuse trades that lower the shadow value of their stocks;

$$T\mathbf{k} \neq \mathbf{k} \to \exists\mathbf{j}, v_j\left((T\mathbf{k})_j\right) > v_j(\mathbf{k}_j): \tag{5}$$

trade takes place only if some firm gains by it; and, where the τth iterate of T is T^τ,

$$(\forall\tau, T^\tau\mathbf{k} = \mathbf{k}) \to \nexists\mathbf{k}' \in Z(\mathbf{k}), \mathbf{v}(\mathbf{k}') \geq \mathbf{v}(\mathbf{k}): \tag{6}$$

trade is permanently foregone only if the gains to trade are exhausted. A $\mathbf{k}$ that satisfies the consequent of (6) I'll call "an exchange equilibrium".

4 Toward exchange equilibrium

Let

$$V(\mathbf{k}) \equiv \sum_j v_j(\mathbf{k}_j), \tag{7}$$

and notice that because $\{T^\tau\mathbf{k}\}_{\tau=0}^\infty$ remains in $Z(\mathbf{k})$ there's a $\{T^{\tau_\upsilon}\mathbf{k}\}_{\upsilon=0}^\infty$ that converges to some $\mathbf{k}^*$. Thanks to the continuity of the exchange map and the value functions,

$$V(\mathbf{k}^*) = V\left(\lim_{\upsilon\to\infty} T^{\tau_\upsilon}\mathbf{k}\right) = \lim_{\upsilon\to\infty} V(T^{\tau_\upsilon}\mathbf{k}). \tag{8}$$

If any such limit point $\mathbf{k}^*$ were not an equilibrium, it would have to be that for some $\mathbf{k}'$

$$\exists\Upsilon : \forall\tau \geq \Upsilon, V(T^\tau\mathbf{k}^*) \geq V(\mathbf{k}') > V(\mathbf{k}^*), \tag{9}$$

a contradiction of the equality of the outermost expressions in (8). So every limit point is an equilibrium, and every trading path approaches the equilibrium set.

5 A strong assumption

With (6) I've made a very strong assumption. (6) holds, for example, that where producer a has the input that's required by a producer b, who has what's needed by c, who has what's needed by d, who has what's needed by e, who has finally what's needed by a, there will come a time when all these goods are passed around this circle. This might happen when T chances to throw all five firms together to consider five-sided trades. Or it might happen via intermediaries. If these firms are unable to store the goods they don't use in production, and if other firms' technologies include the relevant storage activities, then the latter can function as accidental middlemen for whom it will be profitable first to receive these goods from the firms that can't use them and then to pass them along to those who can.

Suppose that (6) is withdrawn. Then there's no proving that exchange approaches the set of states in which gains to trade are exhausted. But a different stability result is still in place. Again there are convergent subsequences $\{T^{\tau_\upsilon}\mathbf{k}\}_{\upsilon=0}^{\infty}$ with limit points $\mathbf{k}^*$. And because V is nondecreasing, no two such limit points can have unequal values of V. And so for some value V^* the system necessarily approaches the set of points in Z for which $V = V^*$.

6 A small open economy?

The argument of section 4 is a trivial development of the convergence results for exchange economies that you have in [1, 5, 8, 11]. But let me point out some complications that have crept in.

The protagonists of those exchange economies are household traders who try to trade their way toward preferred goods holdings. The crux of the stability argument is that the utilities representing those preferences are generally nondecreasing and increasing outside equilibrium. Section 4 arrived at its parallel conclusion by substituting for the households' u's the similarly behaved v's of the firms.

This required that I endow the firms with expectations about the prices at which they'd be able to sell their outputs. And so my argument has left the hardest questions unanswered. Where do these expectations come from? And to whom are the firms expecting to sell the stuff?

The cleanest story you might tell takes as its setting a small open economy. Domestic firms produce goods for sale at the world prices $\mathbf{p}$. And inputs not purchased on the world market circulate within the country according to $T_{\mathbf{p}}$.

Or you might consider a familiar corruption of state socialism. Here $\mathbf{p}$ might stand for the official prices, set by a central planner, at which goods are sold to the state by local enterprises, even as they carry on behind its back an illegal horizontal trade in inputs described by the map $T_{\mathbf{p}}$.

7 Externality and instability

Here's a more abstract view of what's going on. Credit for the easy stability proofs goes to the fact that, in these exchange and production economies, each resource owner depends for her success only on the resources she controls and to the fact that resources change hands only if that's to the advantage of their old and new owners alike. It's because every actor has a veto over the only variables on which her interests depend that those interests can be indexed by the nondecreasing functions through which convergence is so readily demonstrated.

This is not capitalism. In capitalism the capitalist seeks profits that depend on prices that depend on the activity of her fellow capitalists. In the teeth of such interdependence there's no insulating her maximand from factors outside her control. And so there's no hope of proving stability in the manner of section 4.

A few glancing and partial insights can be found here all the same. In the rest of this paper I'll bring a couple of those out.

8 The inefficiency of production organized by exchange

Like the exchange equilibria of the pure-exchange economy, these are Pareto efficient with respect to the interests of the firms: no firm can be made richer in the value of its expected output except by making some firm poorer. They are not however production-efficient in the sense of Koopmans; it's not ruled out that you can increase the aggregate net output of some good at no cost to the production of any other good. For example suppose that one firm's technology is strictly dominated by the production set of another firm. Since the first firm won't trade away its entire input stock, social production is inefficient.

This situation has a dual expression in the prices. The exhaustion of gains to trade implies that firms' shadow prices $\lambda_j(\mathbf{k}_j)$ are necessarily proportional in equilibrium. But it allows that these may differ by a scale factor. Where they do differ in magnitude, social production could be expanded by diverting small quantities of inputs to the firm with the largest $\lambda_j(\mathbf{k}_j)$.

9 Arbitrage

This efficiency gap is closed in a second variant of the model. Let's distinguish a new class, the capitalists, who now own the goods stocks deployed in production at the various firms. The firms are joint ventures among these capitalists, who retain their titles to the inputs they commit and who share the profits in proportion to the values of their contributions. So the cth capitalist who devotes some stocks $\mathbf{k}_{c,j}$ to the jth firm claims a share $\lambda_j(\mathbf{k}_j)\mathbf{k}_{c,j}/\lambda_j(\mathbf{k}_j)\mathbf{k}_j$ of the value of the firm's output $\mathbf{pB}_j\mathbf{x}(\mathbf{k}_j)$. Since $\mathbf{pB}_j\mathbf{x}(\mathbf{k}_j) = \lambda_j(\mathbf{k}_j)\mathbf{k}_j$, the capitalist expects to take $\lambda_j(\mathbf{k}_j)\mathbf{k}_{c,j}$. A myopically profit-maximizing capitalist will try to reapportion her stocks across firms so as to raise the sum of their values in the firms' respective shadow prices.

Suppose that each capitalist encounters opportunities to transfer particular quantities of its stocks to new firms and that it makes such a change if and only if that reinvestment increases the shadow value of its own stocks while conserving the value of its current and new partners' holdings. A reinvesting capitalist can meet the latter condition, if need be, by ceding some of its stock in its old or new deployment to its old or new partners in compensation of any incipient devaluation of their shares. It follows that a given reinvestment opportunity is taken up if and only if this increases the sum of the relevant firm-level programs.

An operator T that represents this process of reallocation will satisfy (3) and (5) as before. I replace the remaining two conditions with

$$V(T\mathbf{k}) \geq V(\mathbf{k}), \tag{10}$$

and

$$(\forall \tau, T^{\tau}\mathbf{k} = \mathbf{k}) \Rightarrow \nexists \mathbf{k}' \in Z(\mathbf{k}), V(\mathbf{k}') > V(\mathbf{k}). \tag{11}$$

The argument of section 3, which used only the behavior of the sum of valuations V, evidently carries over; every trajectory approaches equilibrium, now defined as allocation satisfying the consequent of (11).

10 Quantity and price

Now equilibrium production is Koopmans-efficient: it maximizes the value of aggregate output in the given prices $\mathbf{p}$. So firms' equilibrium shadow prices are necessarily equalized at some value λ^*. And profits as a ratio of inputs valued in those prices are equalized at 1 across all operated activities. If mere exchange among profit-maximizing producers does not suffice for production efficiency and the equalization of prices and profitability, arbitrage under capitalist ownership will bring these about.

Here $V(\mathbf{k})$ comes into its own as a thermodynamic-like potential [cf. 7] the surface of whose maxima picks out the set of macroscopic outcomes to which the system will relax under the particular macroscopic constraints represented by specific values for $\mathbf{k}$. Putting

$$V^*(\mathbf{k}) = \max_{\mathbf{k}' \in Z(\mathbf{k}')} V\left(\mathbf{k}'\right) \tag{12}$$

you have

$$\frac{\partial V^*(\mathbf{k})}{\partial \mathbf{k}} = \lambda^* : \tag{13}$$

the gradient of maximum output with respect to the stocks coincides with the equilibrium input prices. And there's an informative duality between the system's extensive and intensive variables, its aggregate goods stocks $\mathbf{k}$ and its equilibrium prices λ^*.

Consider again the small-open-economy interpretation that I mentioned in section 5. If world prices $\mathbf{p}$ prevail for the traded goods, trade or foreign direct investment in a previously untraded inputs will take place between two such economies if and only if their systems of equilibrium shadow prices differ. So the intensive variables play their usual role of determining whether or not the extensive variables will be exchanged between two initially isolated subsystems [cf. 10].

11 Labor and consumption

Now suppose that production requires labor. Along with the material coefficients $\mathbf{B}_j$ and $\mathbf{A}_j$, the jth capitalist faces unit labor requirements $\mathbf{l}_j$. Claims to labor power available for capitalist production or domestic production or consumption are traded along with produced goods. A working-class household indexed by s starts out with a unit of labor power and no produced goods, and it strives to improve its material position as described by a function $u_s(\mathbf{k}_s, h_s)$ of its physical stocks and of the labor power remaining to it after some portion is sold to capitalists.

With h_j the labor power controlled by the jth firm, let $v_j(\mathbf{k}_j, h_j)$ be the value of the dual programs

$$\max_{\mathbf{x} \in \Re_+^{m(j)}} \mathbf{p}\mathbf{B_j}\mathbf{x} \text{ s.t. } \mathbf{A}_j\mathbf{x} \leqq \mathbf{k}_j,\ \mathbf{l}_j\mathbf{x} \leq h_j \tag{14}$$

and

$$\min_{\lambda\in\Re_{+}^{n},w\in\Re}\lambda\mathbf{k}_j+wh_j \text{ s.t. } \lambda\mathbf{A}_j+w\mathbf{l}_j\geqq\mathbf{pB_j}, \tag{15}$$

let κ represent the capitalists' and workers' stocks of produced goods and claims to labor power, and put

$$\omega(\mathbf{k})\equiv\mathbf{v}(\mathbf{k}_j,h_j),u_1(\mathbf{k}_1,h_1),...u_z(\mathbf{k}_z,h_z). \tag{16}$$

Trade now follows an H that satisfies versions of (3,4,5,6) with H substituted for T, κ for $\mathbf{k}$, and ω for $\mathbf{v}$. The fixed points of H are again trading equilibria that are efficient in the sense that no redistribution raises the value of some actor's program without depressing some other actor's value. Trading paths approach exchange equilibrium, and the values of capitalists' and workers' programs converge to some configuration ω^*.

12 Edgeworth's indeterminacy and Sraffa's

Edgeworth's "indeterminacy of contract" [3] consists here in the fact that, for given initial holdings κ, the asymptotic outcome of trade ω^* depends on which of the many admissible H's obtains. And so it depends on the transactions technology, commercial conventions, competition policy, labor law, collective-bargaining arrangements, and shopfloor balances of power that govern bargaining among capitalists and between capitalists and workers.

Also notice that the values of the workers' and the capitalists' programs are antagonistic across the equilibrium set: the jth capitalist's equilibrium value can increase and all other profit rates remain constant only if the sum of workers' consumption values decreases.

So Edgeworth's theme of indeterminacy encompasses the Marxian or Sraffian observation [9] that distribution is the creature not only of scarcity but of contestable institutions. For a given social endowment of material inputs and human productive power, the asymptotic price system and the asymptotic class income distribution represented by $\omega^*,\lambda^*,\mathbf{w}^*$ depend on the "correlation of class forces" that gives a determinate form to H.

13 Accumulation ongoing

The idealizations on offer so far are sharply limited in their explanatory power by their assumption of given output prices and by their relegation of production to a notional horizon that follows the modelled time of trade. It would be nice to go further.

In the rest of this paper, taking only a first step, I'll suppose that production and the reallocation of inputs across firms unfold on the same time scale. Reallocative opportunities alternate with rounds of production, and outputs are sold as inputs at generally dispersed prices. I will suppose that firms try to maximize their rates of physical expansion, as given by the rates of increase of their most rapidly expanding input stock. This goal is a proxy for maximal accumulation of productive wealth as valued in market prices. A proxy is called for since market prices don't exist outside equilibrium.

My strategy will be to preserve the avowedly artificial feature isolated in section 7 as the guarantor of stability. I'll devise things, then, so that each firm enjoys a final say over the variables that determine its accumulation rate.

14 Contracting for growth

Long-term supply relations should here take the place of one-shot exchange as only the first make it possible to define a long-run equilibrium with trade ongoing. So firms contract with other firms for ongoing deliveries of inputs. A contract concluded at the τth period between the jth and the kth firm calls for net transfers (positive, negative, or zero) of the n commodities from the kth to the jth firm that are proportional to some $\mathbf{z}_{j,k,\tau}$ and that grow like $(1+g_{j,k})^{t-\tau}$.

Between rounds of production firms draw opportunities to negotiate new contracts. Two firms make such a contract if and only if: (i) for each firm there's a sequence of production plans obeying for every t

$$\mathbf{A}_j\mathbf{x}^{t+1} \leqq \mathbf{k}^t = \mathbf{B}_j\mathbf{x}^t + \textstyle\sum_{\tau=1}^{t}\sum_{k\neq j}\mathbf{z}_{j,k,\tau}(1+g_{j,k,\tau})^{t-\tau} \tag{17}$$

so that each can sustainably produce the goods contractually required for delivery; and (ii) the new contract does not decrease the maximum rate of expansion of either firm's input stocks, and for at least one firm it increases this rate, given by

$$g = -1 + \max_{h}\max_{\{\mathbf{x}^t\}}\lim\nolimits_{t\to\infty}\left[\frac{k^t_{j,h}}{k^\tau_{j,h}}\right]^{\tau-t} \quad \text{s.t. (17).} \tag{18}$$

As before I assume that, between production rounds, firms also receive chances to make one-time swaps of given quantities of inputs. And that they do this if and only if no firm's value of g is depressed, at least one firm's g is increased, and both firms can still meet their current contractual commitments.

Finally I assume that firms in every period carry out the production corresponding to the currently maximal g.

15 Toward the long run

Since each firm's problem is homogeneous of degree zero in its current stocks and current contractual net transfer vectors, you can describe the system by confining these to an appropriately dimensioned simplex. And you can confine the rates of delivery expansion to some long-enough interval. And so the system is fully described by a state that stays in a bounded set.

Let G be the greatest value of g economywide. By my assumptions, every firm's value of g is nondecreasing through its supply contracting, production, and exchange activity. And so must be G. And so the limit point of every convergent subsequence must have the same value G^*. Say that a good is *basic* if no good can be produced except by inputs produced by produced by inputs ... that include this good somewhere along the way. Since no good can expand faster than the slowest-growing basic good, the maximum rates of expansion of all basic goods also approach G^*.

While there remain production and contracts that provide for deliveries that grow at rates less than G^*, the stocks tied up in these account for a vanishing share of the aggregate stocks, whose growth rates thus approach G^*. Differential growth complements recontracting and exchange as a form of decentralized reallocation through which the system makes its way toward long-run equilibrium.

16 Maximal growth

If in parallel to my earlier strong assumption you assume that the contracting process will eventually uncover any unexploited opportunity for faster growth by bringing the relevant producers together to contract for deliveries that expand at that higher rate, then the attracting growth rate G^* must be the system's rate of maximal balanced expansion.

As a description of decentralized allocation this assumption is unsatisfying. But once again it's unnecessary for stability. If the firms never manage to coordinate around maximal growth, the system nevertheless approaches a set of production states in which growth is balanced at a less than maximal rate. Albeit perhaps at zero.

17 Long-run prices

Take another look at the constraint (17) on a firm's current-period production:

$$\mathbf{A}_j \mathbf{x}^{t+1} \leqq \mathbf{k}_j^t = \mathbf{B}_j \mathbf{x}^t + \sum_{\tau=1}^{t} \sum_{k \neq j} \mathbf{z}_{j,k,\tau} (1 + g_{j,k,\tau})^{t-\tau}. \tag{19}$$

If any two firms were showing nonproportional Lagrange multipliers on this constraint, then by swapping tranches of their current stocks $\mathbf{k}^t$ they could increase the values of their programs. So these shadow prices are proportional in the long-run equilibrium. Normalize them to some $\mathbf{p}$.

A firm with $\mathbf{p}\mathbf{z}_{k,j} < 0$ in these prices would achieve a higher growth rate by quitting its current contract if it could. So it is not growing at the fastest currently feasible rate. And so in long-run equilibrium it claims a vanishing share of production. Any firms with nonvanishing shares thus have $\mathbf{p}\mathbf{z}_{k,j} \geq 0$. But since $\mathbf{p}\mathbf{z}_{j,k} = -\mathbf{p}\mathbf{z}_{k,j}$ it follows that for these firms

$$\mathbf{p}\mathbf{z}_{j,k} = 0. \tag{20}$$

Multiplying through the constraint by these shadow prices while observing complementary slackness and taking (20) into account, you have that for every sustainable firm

$$\mathbf{p}\mathbf{B}_j \mathbf{x}^t = \mathbf{p}\mathbf{A}_j \mathbf{x}^{t+1} \tag{21}$$

Because production by these firms is following

$$\mathbf{x}^{t+1} = (1 + G^*) \mathbf{x}^t, \tag{22}$$

it turns out that

$$\mathbf{p}\mathbf{B}_j \mathbf{x}^t = (1 + G^*) \mathbf{p}\mathbf{A}_j \mathbf{x}^t \tag{23}$$

In the long run inputs are valued and exchanged at prices that equalize profitability across the active firms at the going rate of accumulation. It's a familiar destination.

18 Thanks

I found out about this kind of decentralized exchange from Duncan Foley, and his ideas shaped the entire paper. (The same goes for nearly every other economic thought I've had in the last 12 years.) I thank Foley along with Rajiv Sethi, Gil Skillman, Peter Skott, Roberto Veneziani, and Luca Zamparelli.

19 References

[1] Robert Axtell, 2005. The complexity of exchange. *Economic journal*
[2] Gérard Debreu, 1970. Economies with a finite set of equilibria. *Econometrica* 38: 387-92.
[3] Francis Ysidro Edgeworth, 1881. *Mathematical psychics.*
[4] Franklin Fisher, 1983. *Disequilibrium foundations of equilibrium economics.* Cambridge University Press
[5] Duncan K. Foley, 2010. What's wrong with the fundamental existence and welfare theorems? *Journal of economic behavior and organization.*
[6] Frank Hahn, 1982. Stability. in Arrow and Intriligator, editors, *Handbook of mathematical economics*, volume 2. Amsterdam: Elsevier.
[7] John von Neumann, 1945. A model of general economic equilibrium. *Review of economic studies*, 33: 1-9.
[8] Steven Smale, 1976. Exchange processes with price adjustment. *Journal of mathematical economics*, 3: 211-226.
[9] Piero Sraffa, 1960. *Production of commodities by means of commodities.* Cambridge.
[10] Eric Smith and Duncan K. Foley, 2008. Classical thermodynamics and economic general equilibrium theory. *Journal of economic dynamics and control.* 32:7-65.
[11] Hirofumi Uzawa, 1962. On the stability of Edgeworth's barter process. *International economic review*, 3: 218-32.

3525 words.